Asymptotically precise norm estimates of scattering from a small circular inhomogeneity

Derek J. Hansen, Clair Poignard, Michael S. Vogelius

Asymptotically precise norm estimates of scattering from a small circular inhomogeneity

Derek J. Hansen, Clair Poignard and Michael S. Vogelius *

Nov 30, 2006

Abstract

We establish L^2-based estimates of the scattering produced by a small circular inhomogeneity. These estimates apply to any frequency, and most importantly they exhibit a behaviour that is consistent with numerically observed solutions, uniformly in frequency and size of the inhomogeneity.

1. Introduction
2. The main estimates for the scattered solution
3. Discussion of extensions and future directions
4. References

1 Introduction

In this paper we shall rigorously estimate the size of the electromagnetic scattering caused by a single small inhomogeneity at all frequencies. For simplicity we restrict our attention to the "transverse magnetic" situation, in which case the scalar electric field satisfies a two dimensional Helmholtz equation. We take the inhomogeneity (or to be quite precise, the cross-section of the inhomogeneity) to be a disk of radius ϵ, but we do believe similar estimates hold for rather arbitrary convex inhomogeneities. The coordinate system is chosen so that the inhomogeneity is centered at the origin. For simplicity we assume that the magnetic permeability equals 1 inside as well as outside the inhomogeneity, but a jump could easily be accommodated. The electric permittivity, q_ϵ, equals q inside the inhomogeneity and q_0 outside; in other words

$$q_\epsilon(y) = \begin{cases} q & \text{for } r = |y| < \epsilon \\ q_0 & \text{for } r = |y| > \epsilon \end{cases} .$$

*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA, and Institut Camille Jordan, UMR CNRS 5208, Université Claude Bernard Lyon 1, 69622 Villeurbanne cedex, France.

We shall assume that the constant q_0 is real and positive. The constant q, the electric permittivity of the inhomogeneity, may be complex, but must have a positive real part and a non-negative imaginary part. In other words we assume that

$$q_0 > 0 \quad , \qquad \Re(q) > 0 \quad , \qquad \text{and} \quad \Im(q) \geq 0 \quad . \tag{1}$$

For some of our estimates to hold we shall require that the inhomogeneity be electrically conducting, i.e., we shall suppose that $\Im(q) > 0$. We note that since q is a constant, independent of ω, this means that the actual physical conductivity of the inhomogeneity (typically equal to $\omega\Im(q)$) depends on ω. The object of study is the solution, u_ϵ, to the equation

$$\Delta u_\epsilon + \omega^2 q_\epsilon u_\epsilon = \quad \text{in } \mathbb{R}^2 \quad , \tag{2}$$

for which the "backscattered" part $u_\epsilon^{(s)}(y) = u_\epsilon(y) - u^{(inc)}(y)$, $|y| > \epsilon$, satisfies the "outgoing" radiation condition

$$\frac{\partial}{\partial r} u_\epsilon^{(s)} - i\omega\sqrt{q_0}u_\epsilon^{(s)} = o(r^{-1/2}) \quad \text{as } r \to \infty \quad . \tag{3}$$

The incident wave, $u^{(inc)}$, is prescribed and satisfies $\Delta u^{(inc)} + \omega^2 q_0 u^{(inc)} = 0$ in $\mathbb{R}^2$. It therefore has the form

$$u^{(inc)}(r,\theta) = \sum_{k=-\infty}^{\infty} a_k J_k(\sqrt{q_0}\omega r)e^{ik\theta} \quad , \tag{4}$$

where $J_k(\cdot)$ denotes the Bessel function of the first kind of order k. We shall always minimally assume that

$$\sum_{k=-\infty}^{\infty} |a_k|^2(1+|k|)^{2\sigma} < \infty \tag{5}$$

for some $\sigma \in \mathbb{R}$. This assumption does allow a "plane wave" of incident direction $\eta = (\cos\theta_0, \sin\theta_0)$:

$$u^{(inc)}(y) = e^{i\omega\sqrt{q_0}\eta\cdot y} = \sum_{k=-\infty}^{\infty} J_k(\sqrt{q_0}\omega r)e^{-ik(\theta_0-\frac{\pi}{2})}e^{ik\theta} \quad , \tag{6}$$

in which case $|a_k| = 1$ for all k. The identity (6) follows directly from symmetry considerations and the integral representation formula

$$J_k(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - k\theta)\, d\theta \quad ,$$

valid for $\Re(z) > 0$, and any integer k. It is well known that, given any $r > 0$, there exist positive constants C_r and c_r such that

$$|J_k(r)| \leq C_r e^{-c_r|k|} \qquad \text{for all } k \in \mathbb{Z} \quad ,$$

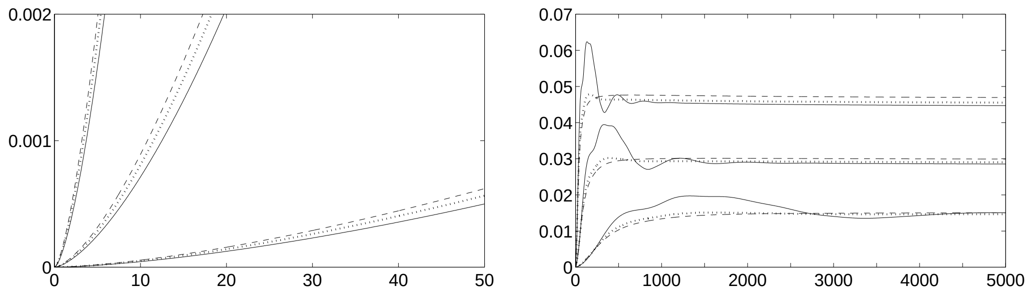


Figure 1: The L^2 norm of $u_\epsilon^{(s)}|_{r=2}$ as a function of ω for three different values of ϵ and three different values of q, as discussed in text.

see for instance [11], page 16. Similar estimates hold for all derivatives of the Bessel functions J_k. A condition of the type (5) for some $\sigma \in \mathbb{R}$ will thus always guarantee the proper summability of the formula (4), and that the sum is indeed a C^∞ function (the latter also follows by elliptic regularity).

The main focus of this paper is to give asymptotically precise norm estimates for the scattered part of the solution (the component $u_\epsilon^{(s)}$) at a fixed distance from the inhomogeneity. As an illustration consider Figure 1. The nine graphs display $\|u_\epsilon^{(s)}\|_{L^2(\{r=2\})}$ as a function of frequency, ω, for three different values of ϵ (.01, .004, and .001) and three different values of q ($4+i$, $3+3i$, and $1+4i$). For better "viewing" the left frame shows the behaviour for relatively small ω, whereas the right frame corresponds to a much wider band of frequencies. The incident wave is taken to be a plane wave. The 9 graphs clearly fall in three groups (each group corresponding to a different value of ϵ, with the $\epsilon = .001$ group at the bottom). In each group the solid graph corresponds to $q = 4+i$, the dotted to $q = 3+3i$, and the dashed to $1+4i$. The fact that the $\|\cdot\|_{L^2}$ norm of $u_\epsilon^{(s)}|_{r=2}$ behaves like ϵ^2, as $\epsilon \to 0$ for fixed ω, is well known [5], [10]. So is the fact that $\|u_\epsilon^{(s)}\|_{L^2(\{r=2\})}$ behaves like $(\epsilon\omega)^2|H_0^{(1)}(\omega 2)|$ when $\epsilon \to 0$, and $\omega = o(\epsilon^{-1})$ [5]. This helps explain the behaviour of these graphs for frequencies ω that are small compared to ϵ^{-1}. In the present paper we shall provide estimates that help explain the behaviour of these graphs for ω of magnitude ϵ^{-1}, and greater. In particular, with varying assumptions about the incident wave, we shall essentially show that $\|u_\epsilon^{(s)}\|_{L^2(\{r=2\})}$ is bounded by $C\sqrt{\epsilon}$, in complete agreement with the right frame of Figure 1. This bound is also consistent with the asymptotic approximation to $u_\epsilon^{(s)}|_{r=2}$ obtained by (formal) techniques of geometric optics in [5].

At the end of this paper we shall briefly discuss some results that may help extend our asymptotic estimates to arbitrary convex inhomogeneities of the form ϵD. We shall also briefly discuss applications of these bounds and their generalizations to problems of inverse scattering.

2 The main estimates for the scattered solution

The problem (2) with its associated outgoing radiation condition may also be written

$$\Delta u_\epsilon^{(t)} + \omega^2 q u_\epsilon^{(t)} = 0, \qquad \text{for } r = |y| < \epsilon \quad , \tag{7a}$$

$$\Delta u_\epsilon^{(s)} + \omega^2 q_0 u_\epsilon^{(s)} = 0, \qquad \text{for } r = |y| > \epsilon \quad , \tag{7b}$$

with the following transmission conditions at $r = \epsilon$:

$$\partial_r u_\epsilon^{(s)}\Big|_{r=\epsilon} = \partial_r u_\epsilon^{(t)}\Big|_{r=\epsilon} - \partial_r u^{(inc)}\Big|_{r=\epsilon} \quad , \tag{7c}$$

$$u_\epsilon^{(s)}\Big|_{r=\epsilon} = u_\epsilon^{(t)}\Big|_{r=\epsilon} - u^{(inc)}\Big|_{r=\epsilon} \quad , \tag{7d}$$

and the asymptotic radiation condition:

$$\partial_r u_\epsilon^{(s)} - i\omega\sqrt{q_0}\, u_\epsilon^{(s)} = o(r^{-1/2}) \qquad \text{as } r \to \infty \quad . \tag{7e}$$

Here we have used the notation $u_\epsilon^{(t)}$ for the solution u_ϵ inside $r < \epsilon$, and we have decomposed the solution in $r > \epsilon$ as

$$u_\epsilon = u_\epsilon^{(s)} + u^{(inc)} \quad .$$

With the assumptions (1) on the constants q_0 and q it is well known that the (transmission) problem (7a)-(7e) has a unique classical solution for an arbitrary smooth incident wave. This follows for instance by arguments (along the lines of those) given in [3] and [6]. For a 2π-periodic function g, we denote by $\widehat{g}_k$ the k^{th} Fourier coefficient of g, defined by

$$\widehat{g}_k = \frac{1}{2\pi}\int_0^{2\pi} g(\theta) e^{-ik\theta}\, d\theta \quad , \qquad k \in \mathbb{Z} \quad .$$

In terms of the Fourier coefficients the equations (7a)-(7e) translate into

$$\frac{d^2}{dr^2}\widehat{u}_{\epsilon,k}^{(t)} + \frac{1}{r}\frac{d}{dr}\widehat{u}_{\epsilon,k}^{(t)} + \left(\omega^2 q - k^2/r^2\right)\widehat{u}_{\epsilon,k}^{(t)} = 0, \qquad r < \epsilon \quad , \tag{8a}$$

$$\frac{d^2}{dr^2}\widehat{u}_{\epsilon,k}^{(s)} + \frac{1}{r}\frac{d}{dr}\widehat{u}_{\epsilon,k}^{(s)} + \left(\omega^2 q_0 - k^2/r^2\right)\widehat{u}_{\epsilon,k}^{(s)} = 0, \qquad r > \epsilon \quad , \tag{8b}$$

with the transmission conditions

$$\frac{d}{dr}\widehat{u}_{\epsilon,k}^{(s)}\Big|_{r=\epsilon} = \frac{d}{dr}\widehat{u}_{\epsilon,k}^{(t)}\Big|_{r=\epsilon} - \frac{d}{dr}\widehat{u}_k^{(inc)}\Big|_{r=\epsilon} \quad , \tag{8c}$$

$$\widehat{u}_{\epsilon,k}^{(s)}\Big|_{r=\epsilon} = \widehat{u}_{\epsilon,k}^{(t)}\Big|_{r=\epsilon} - \widehat{u}_k^{(inc)}\Big|_{r=\epsilon} \quad , \tag{8d}$$

and the radiation condition

$$\frac{d}{dr}\widehat{u}^{(s)}_{\epsilon,k} - i\omega\sqrt{q_0}\widehat{u}^{(s)}_{\epsilon,k} = o(r^{-1/2}) \qquad \text{as } r \to \infty \quad . \tag{8e}$$

The differential equations in (8a)-(8b) are so-called Bessel's equations – the outgoing radiation condition at infinity, (8e), and the fact that $\widehat{u}^{(t)}_{\epsilon,k}$ stays bounded as $r \to 0$ now imply that

$$\widehat{u}^{(s)}_{\epsilon,k}(r) = \alpha_k H^{(1)}_k(\sqrt{q_0}\omega r) \quad , \quad r > \epsilon \quad , \qquad \text{and} \qquad \widehat{u}^{(t)}_{\epsilon,k}(r) = \beta_k J_k(\sqrt{q}\omega r) \quad , \quad r < \epsilon \quad .$$

$H^{(1)}_k$ is a Hankel function of order k. The transmission conditions (8c) and (8d) yield the following identities for the coefficients α_k and β_k

$$\begin{aligned}\alpha_k &\Big\{ \sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J'_k\left(\sqrt{q}\omega\epsilon\right) H^{(1)}_k\left(\sqrt{q_0}\omega\epsilon\right)\Big\} \\ &= a_k\Big\{ \sqrt{q}J'_k\left(\sqrt{q}\omega\epsilon\right) J_k\left(\sqrt{q_0}\omega\epsilon\right) - \sqrt{q_0}J'_k\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right)\Big\} \quad ,\end{aligned}$$

$$\begin{aligned}\beta_k &\Big\{ \sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J'_k\left(\sqrt{q}\omega\epsilon\right) H^{(1)}_k\left(\sqrt{q_0}\omega\epsilon\right)\Big\} \\ &= a_k\Big\{ \sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q_0}\omega\epsilon\right) - \sqrt{q_0}J'_k\left(\sqrt{q_0}\omega\epsilon\right) H^{(1)}_k\left(\sqrt{q_0}\omega\epsilon\right)\Big\} \quad ,\end{aligned}$$

where a_k are the coefficients from the incident wave expression (4). We observe that

Lemma 1. *Suppose q_0 and q satisfy (1). Then*

$$\sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}r\right) J_k\left(\sqrt{q}r\right) - \sqrt{q}J'_k\left(\sqrt{q}r\right) H^{(1)}_k\left(\sqrt{q_0}r\right) \neq 0 \quad ,$$

for any $k \in \mathbb{Z}$, and $r > 0$.

Proof. This non-degeneracy may be deduced from the general uniqueness and exitence result for the Helmholtz equation (2) with a prescribed incident wave, and an outgoing radiation condition, mentioned earlier. However, it also has a very direct and simple proof: if for some k and $r_0 > 0$

$$\sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}r_0\right) J_k\left(\sqrt{q}r_0\right) - \sqrt{q}J'_k\left(\sqrt{q}r_0\right) H^{(1)}_k\left(\sqrt{q_0}r_0\right) = 0 \quad ,$$

then, either $J_k(\sqrt{q}r_0) \neq 0$, and

$$U(r,\theta) = \begin{cases} \frac{H^{(1)}_k(\sqrt{q_0}r_0)}{J_k(\sqrt{q}r_0)} J_k(\sqrt{q}r)e^{ik\theta} & r < r_0 \quad , \\ H^{(1)}_k(\sqrt{q_0}r)e^{ik\theta} & r > r_0 \quad , \end{cases}$$

or $J_k'(\sqrt{q}r_0) \neq 0$, and

$$U(r,\theta) = \begin{cases} \frac{\sqrt{q_0}(H_k^{(1)})'(\sqrt{q_0}r_0)}{\sqrt{q}J_k'(\sqrt{q}r_0)} J_k(\sqrt{q}r)e^{ik\theta} & r < r_0 \quad , \\ H_k^{(1)}(\sqrt{q_0}r)e^{ik\theta} \quad r > r_0 \quad , \end{cases}$$

is a solution to

$$\Delta U + q^* U = 0 \quad \text{in} \quad \mathbb{R}^2 \quad , \tag{9}$$

with

$$\partial_r U - i\sqrt{q_0}U = o(r^{-1/2}) \quad , \quad \text{and} \quad U = O(r^{-1/2}) \quad , \quad \text{as} \quad r \to \infty \quad . \tag{10}$$

Here q^* is given by

$$q^*(y) = \begin{cases} q & \text{for } r = |y| < r_0 \\ q_0 & \text{for } r = |y| > r_0 \end{cases} \quad .$$

Multiplication of (9) by $\overline{U}$ and integration by parts, using (10), now gives

$$\lim_{R\to\infty} \left[-\int_{|y|<R} |\nabla U|^2 \; dy + \int_{|y|<R} q^*|U|^2 \; dy + i\sqrt{q_0} \int_{|y|=R} |U|^2 \; d\sigma_y \right] = 0 \quad ,$$

and so

$$\lim_{R\to\infty} \int_{|y|=R} |U|^2 \; d\sigma_y = -\frac{\Im(q)}{\sqrt{q_0}} \int_{|y|<r_0} |U|^2 \; dy \leq 0 \quad .$$

On the other hand, based on the asymptotics of the Hankel function $H_k^{(1)}$, namely

$$H_k^{(1)}(r) = \sqrt{\frac{2}{\pi r}} e^{i(r - \frac{k\pi}{2} - \frac{\pi}{4})} + O(r^{-3/2}) \quad , \qquad \text{as } r \to \infty \quad ,$$

we easily calculate that

$$\lim_{R\to\infty} \int_{|y|=R} |U|^2 \; d\sigma_y = \frac{4}{\sqrt{q_0}} \quad ,$$

and so we have obviously reached a contradiction. We conclude that the nondegeneracy statement of this lemma must hold. □

Based on this non-degeneracy lemma and the previous identities for α_k and β_k we conclude that

$$\alpha_k = a_k \frac{\sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) J_k\left(\sqrt{q_0}\omega\epsilon\right) - \sqrt{q_0}J_k'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right)}{\sqrt{q_0}(H_k^{(1)})'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)} \quad , \tag{11}$$

$$\begin{aligned} \beta_k &= a_k \frac{\sqrt{q_0}(H_k^{(1)})'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q_0}\omega\epsilon\right) - \sqrt{q_0}J_k'\left(\sqrt{q_0}\omega\epsilon\right) H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)}{\sqrt{q_0}(H_k^{(1)})'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)} \qquad (12) \\ &= a_k \frac{2i}{\pi\omega\epsilon\left[\sqrt{q_0}(H_k^{(1)})'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)\right]} \quad , \end{aligned}$$

where we have used a well-known formula for the Wronskian of the functions J_k and $H_k^{(1)}$ to obtain the last identity. In terms of the α_k and the β_k we have

$$u_\epsilon^{(t)}(y) = \sum_{k=-\infty}^{\infty} \beta_k J_k(\sqrt{q}\omega r)e^{ik\theta} \quad \text{for } r = |y| < \epsilon \ , \tag{13}$$

$$u_\epsilon^{(s)}(y) = \sum_{k=-\infty}^{\infty} \alpha_k H_k^{(1)}(\sqrt{q_0}\omega r)e^{ik\theta} \quad \text{for } r = |y| > \epsilon \ . \tag{14}$$

To estimate the coefficients α_k and β_k as functions of $\omega\epsilon$, for any fixed k, the following two observations about the asymptotic behaviour of Bessel (and Hankel) functions will prove essential.

Lemma 2. *For any integer $k \geq 0$*

$$\sqrt{q_0}(H_k^{(1)})'(\sqrt{q_0}r)J_k(\sqrt{q}r) - \sqrt{q}J_k'(\sqrt{q}r)H_k^{(1)}(\sqrt{q_0}r) = \frac{2i}{\pi}\left(\frac{q}{q_0}\right)^{\frac{k}{2}} r^{-1} \ + \ o(r^{-1}) \ ,$$

and

$$\sqrt{q}J_k'(\sqrt{q}r)J_k(\sqrt{q_0}r) - \sqrt{q_0}J_k'(\sqrt{q_0}r)J_k(\sqrt{q}r) = C_k(qq_0)^{\frac{k}{2}}(q_0-q)r^{2k+1} + o(r^{2k+1}) \ ,$$

as $r \to 0_+$. The constant C_k is given by $C_k = \frac{1}{k!(k+1)!}2^{-(2k+1)}$.

Proof. The identities are straightforward consequences of very well known asymptotic properties of Bessel functions and their derivatives, as found for instance in [11]. We omit the details of the derivation. □

Lemma 3. *Let $c = a + ib$ be a fixed complex constant with $b > 0$. Then, for any integer $k \geq 0$*

$$\begin{aligned} (H_k^{(1)})'(r)J_k(cr) &= i\frac{e^{br}e^{i(1-a)r}}{\pi r\sqrt{c}}\left(1 + O(r^{-1})\right) \ , \\ J_k'(cr)H_k^{(1)}(r) &= -i\frac{e^{br}e^{i(1-a)r}}{\pi r\sqrt{c}}\left(1 + O(r^{-1})\right) \ , \end{aligned}$$

as $r \to \infty$. Furthermore, there exist constants $C_{k,c}$ such that

$$\begin{aligned} |J_k'(r)J_k(cr)| &\leq C_{k,c}\frac{e^{br}}{r} \ , \quad \textit{and} \\ |J_k'(cr)J_k(r)| &\leq C_{k,c}\frac{e^{br}}{r} \ , \end{aligned}$$

for all $r > 0$.

Proof. These identities and bounds follow from known asymptotic behaviour of Bessel functions. Since this particular asymptotic behaviour may be slightly less well known than that which led to Lemma 2, we provide the details of the derivation of the first identity and the first bound. For reasons of brevity we leave the (similar) derivation of the two remaining statements to the reader. We have

$$(H_k^{(1)})'(r)J_k(cr) = \frac{1}{2}\left(H_{k-1}^{(1)}(r) - H_{k+1}^{(1)}(r)\right) J_k(cr) \quad , \tag{15}$$

for $k \geq 0$. As $r \to \infty$ we also have

$$H_k^{(1)}(r) = \left(\frac{2}{\pi r}\right)^{1/2} e^{i(r-\frac{k}{2}\pi-\frac{\pi}{4})}\left(1 + O(r^{-1})\right) \quad , \tag{16}$$

see [8] page 122, or [11] page 198. Insertion of (16) into (15) gives

$$\begin{aligned}(H_k^{(1)})'(r)J_k(cr) &= \frac{1}{2}\left(\frac{2}{\pi r}\right)^{1/2} e^{i(r-\frac{k}{2}\pi-\frac{\pi}{4})}\Bigg(e^{i\frac{\pi}{2}}\left(1+O(r^{-1})\right) \\ &\qquad -e^{-i\frac{\pi}{2}}\left(1+O(r^{-1})\right)\Bigg)J_k(cr) \\ &= i\left(\frac{2}{\pi r}\right)^{1/2} e^{i(r-\frac{k}{2}\pi-\frac{\pi}{4})}\left(1+O(r^{-1})\right)J_k(cr) \quad ,\end{aligned} \tag{17}$$

for $k \geq 0$. Due to the fact that $c = a + ib$ has a positive imaginary part,

$$J_k(cr) = \left(\frac{2}{\pi c r}\right)^{1/2} \frac{e^{-i(cr-\frac{k}{2}\pi-\frac{\pi}{4})}}{2}\left(1 + O(r^{-1})\right) \quad ,$$

and upon insertion of this into (17) we now get

$$\begin{aligned}(H_k^{(1)})'(r)J_k(cr) &= i\frac{1}{\pi r\sqrt{c}}\, e^{i(1-c)r}\,\left(1+O(r^{-1})\right) \\ &= i\frac{e^{br}e^{i(1-a)r}}{\pi r\sqrt{c}}\,\left(1+O(r^{-1})\right) \quad \text{as } r \to \infty \quad ,\end{aligned}$$

for any $k \geq 0$. This verifies the first of the asymptotic identities. We now proceed to verify the first of the inequalities. To this end we have

$$J_k'(r)J_k(cr) = \frac{1}{2}\left(J_{k-1}(r) - J_{k+1}(r)\right)J_k(cr) \quad , \tag{18}$$

for $k \geq 0$. From [11] page 199-201 we get

$$|J_k(z)| \leq C_k\left(\frac{2}{\pi|z|}\right)^{1/2} e^{|\Im(z)|} \quad ,$$

for $|z|$ large; the smoothness of J_k at 0 guarantees that the inequality holds for all z. Insertion of this estimate into (18) immediately yields

$$\begin{aligned} |J_k'(r)J_k(cr)| &\leq C_k \left(\frac{2}{\pi r}\right)^{1/2} \left(\frac{2}{\pi |c| r}\right)^{1/2} e^{|\Im(cr)|} \\ &\leq C_{k,c}\frac{e^{br}}{r} \quad , \end{aligned}$$

as desired. ☐

The above two lemmas, and the formulas (11)–(12) lead to the following estimates for α_k and β_k as functions of $\omega\epsilon$, for any fixed integer k.

Lemma 4. *Suppose q_0 and q satisfy (1). Given any fixed $k \in \mathbb{Z}$ there exists a constant $D_k = D_k(q, q_0)$, independent of $\omega\epsilon$ and a_k, such that for all $0 < \omega\epsilon \leq 1$:*

$$|\alpha_k| \leq D_k|a_k|(\omega\epsilon)^{2|k|+2} \quad , \quad \text{and} \quad |\beta_k| \leq D_k|a_k| \quad . \tag{19}$$

Furthermore, if $\Im(q) > 0$ then $D_k = D_k(q, q_0)$ may be selected so that

$$|\alpha_k| \leq D_k|a_k| \quad , \quad \text{and} \quad |\beta_k| \leq D_k|a_k|e^{-\omega\epsilon\Im(\sqrt{q})} \quad , \tag{20}$$

for all $1 < \omega\epsilon$.

Proof. Since $J_{-k} = (-1)^k J_k$ and $H^{(1)}_{-k} = (-1)^k H^{(1)}_k$ it clearly suffices to verify the statements of this lemma for any $k \geq 0$. From Lemma 2 it follows that

$$\begin{aligned} &\frac{\sqrt{q}J_k'(\sqrt{q}\omega\epsilon)J_k(\sqrt{q_0}\omega\epsilon) - \sqrt{q_0}J_k'(\sqrt{q_0}\omega\epsilon)J_k(\sqrt{q}\omega\epsilon)}{\sqrt{q_0}(H^{(1)}_k)'(\sqrt{q_0}\omega\epsilon)J_k(\sqrt{q}\omega\epsilon) - \sqrt{q}J_k'(\sqrt{q}\omega\epsilon)H^{(1)}_k(\sqrt{q_0}\omega\epsilon)} \\ &\qquad = -\frac{i\pi}{2}C_k q_0^k(q_0 - q)(\omega\epsilon)^{2k+2} \ + \ o((\omega\epsilon)^{2k+2}) \quad , \quad \text{and} \\ &\frac{2i}{\pi\omega\epsilon\left[\sqrt{q_0}(H^{(1)}_k)'(\sqrt{q_0}\omega\epsilon)J_k(\sqrt{q}\omega\epsilon) - \sqrt{q}J_k'(\sqrt{q}\omega\epsilon)H^{(1)}_k(\sqrt{q_0}\omega\epsilon)\right]} \\ &\qquad = \left(\frac{q_0}{q}\right)^{k/2} \ + \ o(1) \quad , \end{aligned} \tag{21}$$

as $\omega\epsilon \to 0$. Due to the continuity of these two expressions with respect to $\omega\epsilon \in (0, 1]$ (the nondegeneracy assured by Lemma 1) and the formulas (11)-(12) it follows that

$$|\alpha_k| \leq D_k|a_k|(\omega\epsilon)^{2|k|+2} \quad , \quad \text{and} \quad |\beta_k| \leq D_k|a_k| \quad ,$$

for any $0 < \omega\epsilon \leq 1$. We now turn to the last estimates of this lemma. Lemma 3, with $c = \sqrt{q}/\sqrt{q_0}$ and $r = \sqrt{q_0}\omega\epsilon$, implies that

$$\begin{aligned} &\sqrt{q_0}(H^{(1)}_k)'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) H^{(1)}_k\left(\sqrt{q_0}\omega\epsilon\right) \\ &\qquad = \frac{i}{\pi\omega\epsilon}\frac{\sqrt{q_0} + \sqrt{q}}{(q_0 q)^{1/4}} e^{\omega\epsilon\Im(\sqrt{q})} e^{i(\sqrt{q_0} - \Re(\sqrt{q}))\omega\epsilon}\left(1 + O((\omega\epsilon)^{-1})\right) \quad , \end{aligned}$$

as $\omega\epsilon \to \infty$, and

$$\left|\sqrt{q} J_k' \left(\sqrt{q}\omega\epsilon\right) J_k \left(\sqrt{q_0}\omega\epsilon\right) - \sqrt{q_0} J_k' \left(\sqrt{q_0}\omega\epsilon\right) J_k \left(\sqrt{q}\omega\epsilon\right)\right| \leq C_k e^{\omega\epsilon\Im(\sqrt{q})} (\omega\epsilon)^{-1} \quad ,$$

with C_k depending only on k (and q_0 and q). As a consequence of these two estimates, the formula (11) (and the fact that the fraction in the right hand side of (11) depends smoothly on $\omega\epsilon \in [1, \infty)$) we get

$$|\alpha_k| \leq D_k |a_k| \quad , \qquad \text{for } 1 < \omega\epsilon \quad ,$$

with D_k only dependent on k (and q_0 and q) but independent of ω and ϵ. By a similar argument it follows that

$$|\beta_k| \leq D_k |a_k| e^{-\omega\epsilon\Im(\sqrt{q})} \quad .$$

This completes the proof of the lemma. □

Lemma 4 enables us to prove the following result for the scattered solution corresponding to an incident wave with finitely many Fourier coefficients.

Proposition 1. *Suppose q_0 and q satisfy (1). Let $u^{(inc)}$ be an incident wave, given by*

$$u^{(inc)}(r, \theta) = \sum_{|k| \leq k_0} a_k J_k \left(\sqrt{q_0}\omega r\right) e^{ik\theta} \quad ,$$

for some integer $k_0 \geq 0$, and some coefficients a_k, $-k_0 \leq k \leq k_0$. We denote by $|a|_\infty$ the maximum of the moduli of the coefficients a_k. For $\omega\epsilon > 0$ let $u_\epsilon^{(t)}$ and $u_\epsilon^{(s)}$ be the solutions to the following two dimensional scattering problem

$$\begin{aligned} \Delta u_\epsilon^{(t)} + q\omega^2 u_\epsilon^{(t)} &= 0 \qquad \textit{for } r = |y| < \epsilon \quad , \\ \Delta u_\epsilon^{(s)} + q_0\omega^2 u_\epsilon^{(s)} &= 0 \qquad \textit{for } r = |y| > \epsilon \quad , \end{aligned}$$

with the transmission conditions

$$\begin{aligned} \partial_r u_\epsilon^{(s)}\Big|_{r=\epsilon} &= \partial_r u_\epsilon^{(t)}\Big|_{r=\epsilon} - \partial_r u^{(inc)}\Big|_{r=\epsilon} \quad , \\ u_\epsilon^{(s)}\Big|_{r=\epsilon} &= u_\epsilon^{(t)}\Big|_{r=\epsilon} - u^{(inc)}\Big|_{r=\epsilon} \quad , \end{aligned}$$

and the outgoing radiation condition

$$\partial_r u_\epsilon^{(s)} - i\omega\sqrt{q_0} u_\epsilon^{(s)} = o(r^{-1/2}) \quad \textit{as } r \to \infty \quad .$$

There exists a constant $C = C_{k_0}$, depending only on k_0 (and q, q_0) but independent of ω, ϵ, and R, such that, for any $R \geq \epsilon$

$$\left\| u_\epsilon^{(s)}\big|_{r=R} \right\|_{L^2(0,2\pi)} \leq C_{k_0} |a|_\infty \begin{cases} (\omega\epsilon)^2 \left|H_0^{(1)}(\omega R)\right| & \textit{for } 0 < \omega\epsilon \leq 1 \quad , \\ 1/\sqrt{\omega R} & \textit{for } 1 < \omega\epsilon \quad . \end{cases} \tag{22}$$

Proof. At this place we only provide a proof under the additional assumption that $\Im(q) > 0$. The arguments necessary to include the case $\Im(q) = 0$ (an improved version of Lemma 4) are presented later in Remark 4. The function $u_\epsilon^{(s)}(y)$, $|y| = r > \epsilon$, is given by

$$u_\epsilon^{(s)}(r,\theta) = \sum_{|k|\le k_0} \alpha_k H_k^{(1)}(\sqrt{q_0}\omega r)e^{ik\theta} \quad , \tag{23}$$

where the coefficients α_k are as before. Due to the first estimate of Lemma 4, and the asymptotics of the Hankel function $H_k^{(1)}$ we get for any $k \neq 0$

$$\left|\alpha_k H_k^{(1)}(\sqrt{q_0}\omega\epsilon)\right| \le D_k |a_k| (\omega\epsilon)^{|k|+2} \le D_k |a_k| (\omega\epsilon)^3 \quad , \quad \text{for } 0 < \omega\epsilon \le 1 \quad .$$

For $k = 0$ we have

$$|\alpha_0| \le D_0 |a_0| (\omega\epsilon)^2 \quad , \quad \text{for } 0 < \omega\epsilon \le 1 \quad .$$

Here D_k is a constant that depends only on k, q and q_0, but is independent of ω and ϵ. We may thus estimate

$$\begin{aligned}
\|u_\epsilon^{(s)}\big|_{r=R}\|^2_{L^2(0,2\pi)} &= 2\pi\,|\alpha_0|^2 \left|H_0^{(1)}(\sqrt{q_0}\omega R)\right|^2 \\
&\quad + 2\pi \sum_{0<|k|\le k_0} |\alpha_k H_k^{(1)}(\sqrt{q_0}\omega\epsilon)\,|^2 \left|\frac{H_k^{(1)}(\sqrt{q_0}\omega R)}{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)}\right|^2 \\
&\le C_{k_0} |a|^2_\infty (\omega\epsilon)^4 \left(\left|H_0^{(1)}(\sqrt{q_0}\omega R)\right|^2 + (\omega\epsilon)^2 \frac{\epsilon}{R}\right) \\
&\le C_{k_0} |a|^2_\infty (\omega\epsilon)^4 \left|H_0^{(1)}(\omega R)\right|^2 \quad ,
\end{aligned}$$

for $0 < \omega\epsilon \le 1$, $\epsilon \le R$. Here we have used a well known fact about Hankel functions, namely that

$$\frac{|H_k^{(1)}(r)|^2}{|H_k^{(1)}(s)|^2} \le \frac{s}{r} \quad , \qquad \text{for} \quad 0 < s \le r \quad , \quad k \neq 0 \quad . \tag{24}$$

The estimate (24) follows from the fact that the function $r \to r\left|H_k^{(1)}(r)\right|^2$ is a *decreasing* function on $(0, \infty)$ (cf. [11] pg. 446) for any integer $k \neq 0$. We have also used that

$$|H_0^{(1)}(\sqrt{q_0}\omega R)\,|^2 \le C|H_0^{(1)}(\omega R)\,|^2 \quad ,$$

and that

$$\begin{aligned}
(\omega\epsilon)^2 \frac{\epsilon}{R} &\le \min\{(\omega R)^2, \frac{1}{\omega R}\} \\
&\le C|H_0^{(1)}(\omega R)\,|^2 \quad , \quad \text{for } 0 < \omega\epsilon \le 1 \quad , \quad \epsilon \le R \quad .
\end{aligned}$$

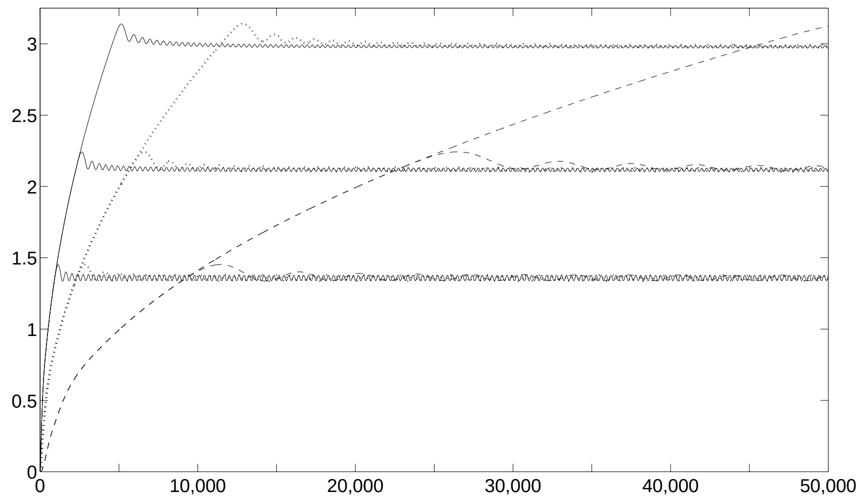


Figure 2: $\sqrt{\omega}\|u_\epsilon^{(s)}|_{r=2}\|_{L^2}$ as a function of ω for three different values of ϵ and three different numbers of Fourier coefficients, k_0, of the incident wave.

The first of these estimates follows from the fact that neither of the functions $|H_0^{(1)}\left(\sqrt{q_0}\,\cdot\right)|$ and $|H_0^{(1)}(\cdot)|$ vanish on $(0,\infty)$, and the fact that they are asymptotically equivalent for small, as well as for large positive values. The second estimate is a consequence of well known asymptotic behaviour of Hankel functions. Altogether we have now proven the estimate (22) for $\omega\epsilon \in (0,1]$. In order to complete the proof of this proposition it thus remains to prove that

$$\|u_\epsilon^{(s)}\big|_{r=R}\|^2_{L^2(0,2\pi)} \le C_{k_0}\frac{|a|^2_\infty}{\omega R} \quad , \qquad \omega\epsilon \in (1,\infty) \quad . \tag{25}$$

A combination of the last α_k estimate of Lemma 4 and the representation formula (23) yields

$$\begin{aligned}\|u_\epsilon^{(s)}\big|_{r=R}\|^2_{L^2(0,2\pi)} &= 2\pi \sum_{|k|\le k_0} |\alpha_k|^2 |H_k^{(1)}(\sqrt{q_0}\omega R)|^2 \\ &\le C_{k_0}|a|^2_\infty(\omega R)^{-1} \quad ,\end{aligned}$$

for $1 < \omega\epsilon$. Here we have used the well known fact that $|H_k^{(1)}(r)|^2 \le C_k r^{-1}$ for $0 < r_0 < r$. This completes the proof of (25), and thus the proof of the proposition, in the case $\Im(q) > 0$. $\square$

Remark 1. *The optimality of the bottom estimate of (22) is illustrated by Figure 2, where we display the rescaled L^2 norm $\sqrt{\omega}\|u_\epsilon^{(s)}|_{r=2}\|_{L^2}$ as a function of ω for* *three different values of ϵ, and for three different "truncated" incident plane waves*

$$u^{(inc)}(y) = \sum_{|k|\le k_0} J_k(\sqrt{q_0}\omega r)e^{ik\theta} \quad .$$

with $k_0 = 10,\ 25,$ and 50, respectively. The dashed curves correspond to $\epsilon = .001$, the dotted curves to $\epsilon = .004$, and the solid curves to $\epsilon = .01$. For the calculations shown in Figure 2 we have chosen $q_0 = 1$, $q = 2 + 2i$. Note that for a fixed ϵ the curves corresponding to the two smallest values of k_0 (10 and 25) significantly "bifurcate" from the curve corresponding to $k_0 = 50$ at frequencies ω with $\omega\epsilon$ of the size 10 and 25, respectively. Also note that the asymptotic value of $\sqrt{\omega}\|u_\epsilon^{(s)}|_{r=2}\|_{L^2}$ first seems to be achieved at a frequency for which $\omega\epsilon$ is of the size k_0.

Remark 2. *The top estimate of (22) also holds for incident waves with infinite Fourier series, at least in the case when $\omega\epsilon \to 0$, i.e., $\omega = o(\epsilon^{-1})$. An outline of a proof of this is found in [5]. As seen from the right frame of Figure 1 the bottom estimate of (22) does not hold in general for incident waves with infinite Fourier series. The bottom estimate of (22) does imply that*

$$\left\| u_\epsilon^{(s)}|_{r=R} \right\|_{L^2(0,2\pi)} \le C_{k_0}|a|_\infty \frac{1}{\sqrt{\omega R}} = C_{k_0}|a|_\infty \frac{1}{\sqrt{\omega\epsilon}}\frac{\sqrt{\omega\epsilon}}{\sqrt{\omega R}} \le C_{k_0}|a|_\infty \frac{\sqrt{\epsilon}}{\sqrt{R}} \quad ,$$

for $1 < \omega\epsilon$. We conjecture that this weaker estimate

$$\left\| u_\epsilon^{(s)}|_{r=R} \right\|_{L^2(0,2\pi)} \le C|a|_\infty \frac{\sqrt{\epsilon}}{\sqrt{R}} \quad , \qquad 1 < \omega\epsilon \quad ,$$

holds even for incident waves with infinite Fourier series. The next lemma and the following proposition verifies a modified version of this conjecture.

In order to estimate the size of $u_\epsilon^{(s)}$, for incident waves with infinitely many Fourier coefficients and for large $\omega\epsilon$, we need estimates for the coefficients α_k that are valid uniformly in $k \in \mathbb{Z}$ and $1 < \omega\epsilon$. Whereas Proposition 1 holds for $\Im(q) = 0$ (see Remark 4 later in this section) it is essential for the approach taken here that the inhomogeneity be electrically conducting (*i.e.*, that $\Im(q) > 0$) when dealing with incident waves with infinite Fourier series.

Lemma 5. *Suppose q_0 and q satisfy (1). There exists a positive constant c, depending on q, but independent of q_0, k, ω, and ϵ, such that*

$$\left| \sqrt{q_0}\ (H_k^{(1)})'\left(\sqrt{q_0}\omega\epsilon\right) J_k\left(\sqrt{q}\omega\epsilon\right) - \sqrt{q}J_k'\left(\sqrt{q}\omega\epsilon\right) H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right) \right|^2 \ge c\Im(q)\frac{1+|k|}{(1+\omega\epsilon+|k|)^2}|J_k(\sqrt{q}\omega\epsilon)|^2 \quad , \qquad (26)$$

for any $k \in \mathbb{Z}$ and any $\omega\epsilon > 0$. Suppose additionally $\Im(q) > 0$, and let α_k and β_k denote the coefficients from the expansion (13)-(14) of u_ϵ. Then there exists *a constant C, depending on q, but independent of q_0, k, ω, and ϵ, such that*

$$
\begin{aligned}
|\alpha_k| &\leq C|a_k|\left(\sqrt{1+|k|}+|J_k(\sqrt{q_0}\omega\epsilon)|\right)|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)|^{-1}, &(27)\\
|\beta_k| &\leq C|a_k|\sqrt{1+|k|}\;|J_k(\sqrt{q}\omega\epsilon)|^{-1}, &(28)
\end{aligned}
$$

for all $k \in \mathbb{Z}$ and $1 < \omega\epsilon$.

Proof. A well known identity for the Wronskian of J_k and Y_k (*cf.* [8] page 113) yields that

$$
\begin{aligned}
&\Im\left(\sqrt{q_0}\;(H_k^{(1)})'\,(\sqrt{q_0}\omega\epsilon)\,\overline{H_k^{(1)}\,(\sqrt{q_0}\omega\epsilon)}\right)\\
&\qquad= \sqrt{q_0}\left(J_k\,(\sqrt{q_0}\omega\epsilon)\,Y_k'\,(\sqrt{q_0}\omega\epsilon) - J_k'\,(\sqrt{q_0}\omega\epsilon)\,Y_k\,(\sqrt{q_0}\omega\epsilon)\right)\\
&\qquad= \frac{2}{\pi\omega\epsilon}\quad. \qquad (29)
\end{aligned}
$$

Green's formula, and the fact that $(\Delta + q(\omega\epsilon)^2)\left[J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right] = 0$, gives

$$
\begin{aligned}
\sqrt{q}J_k'(\sqrt{q}\omega\epsilon)\overline{J_k(\sqrt{q}\omega\epsilon)} &= \frac{1}{\omega\epsilon}\left(\frac{d}{dr}\bigg|_{r=1} J_k(\sqrt{q}\omega\epsilon r)\right)\overline{J_k(\sqrt{q}\omega\epsilon)}\\
&= \frac{1}{2\pi\omega\epsilon}\|\nabla\left(J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right)\|^2_{L^2(\{|y|<1\})}\\
&\quad+\frac{1}{2\pi\omega\epsilon}\int_{|y|<1}\Delta\left[J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right]\overline{J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}}\,dy\\
&= \frac{1}{2\pi\omega\epsilon}\|\nabla\left(J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right)\|^2_{L^2(\{|y|<1\})}\\
&\quad-\frac{q\omega\epsilon}{2\pi}\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{L^2(\{|y|<1\})}\quad. \qquad (30)
\end{aligned}
$$

Let $T(q_0, q, k, \omega\epsilon)$ denote the expression

$$
T(q_0,q,k,\omega\epsilon) = \sqrt{q_0}\;(H_k^{(1)})'\,(\sqrt{q_0}\omega\epsilon)\,J_k\,(\sqrt{q}\omega\epsilon) - \sqrt{q}J_k'\,(\sqrt{q}\omega\epsilon)\,H_k^{(1)}\,(\sqrt{q_0}\omega\epsilon)\quad.
$$

A simple calculation yields

$$
\begin{aligned}
&T(q_0,q,k,\omega\epsilon)\overline{J_k\,(\sqrt{q}\omega\epsilon)}\overline{H_k^{(1)}\,(\sqrt{q_0}\omega\epsilon)}\\
&\qquad= \sqrt{q_0}\;(H_k^{(1)})'(\sqrt{q_0}\omega\epsilon)\overline{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)}\,|J_k(\sqrt{q}\omega\epsilon)|^2\\
&\qquad\quad-\sqrt{q}J_k'(\sqrt{q}\omega\epsilon)\overline{J_k(\sqrt{q}\omega\epsilon)}\left|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)\right|^2\quad,
\end{aligned}
$$

and by combining with (29) and (30) we now obtain

$$
\begin{aligned}
&\Im\left(T(q_0,q,k,\omega\epsilon)\overline{J_k\,(\sqrt{q}\omega\epsilon)}\overline{H_k^{(1)}\,(\sqrt{q_0}\omega\epsilon)}\right) \qquad (31)\\
&\quad= \frac{2}{\pi\omega\epsilon}\,|J_k(\sqrt{q}\omega\epsilon)|^2 + \frac{\Im(q)\omega\epsilon}{2\pi}\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{L^2(\{|y|<1\})}\left|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)\right|^2.
\end{aligned}
$$

We also estimate

$$\begin{aligned}
&\Im\left(T(q_0,q,k,\omega\epsilon)\overline{J_k\left(\sqrt{q}\omega\epsilon\right)}\overline{H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)}\right)\\
&\qquad\leq\left|T(q_0,q,k,\omega\epsilon)\overline{H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)}\overline{J_k\left(\sqrt{q}\omega\epsilon\right)}\right| \qquad (32)\\
&\qquad\leq\frac{\pi\omega\epsilon}{8}\left|T(q_0,q,k,\omega\epsilon)\right|^2\left|H_k^{(1)}\left(\sqrt{q_0}\omega\epsilon\right)\right|^2+\frac{2}{\pi\omega\epsilon}\left|J_k\left(\sqrt{q}\omega\epsilon\right)\right|^2 \quad .
\end{aligned}$$

A combination of (31) and (32) leads to

$$|T(q_0,q,k,\omega\epsilon)|^2\geq\frac{4\Im(q)}{\pi^2}\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{L^2(\{|y|<1\})} \quad . \qquad (33)$$

Since

$$\|\Delta\left(J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right)\|_{L^2(\{|y|<1\})}=|q|(\omega\epsilon)^2\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{L^2(\{|y|<1\})} \quad ,$$

we get, by elliptic regularity estimates,

$$\begin{aligned}
&\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{H^2(\{|y|<1\})}\\
&\quad\leq C\left(\|\Delta\left(J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\right)\|_{L^2(\{|y|<1\})}+\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{H^{3/2}(\{|y|=1\})}\right)\\
&\quad\leq C\left((\omega\epsilon)^2\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{L^2(\{|y|<1\})}+(1+|k|)^{3/2}|J_k(\sqrt{q}\omega\epsilon)|\right) \quad .
\end{aligned}$$

Due to the logarithmic convexity of the Sobolev norms it follows that

$$\begin{aligned}
&\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{H^1(\{|y|<1\})}\\
&\qquad\leq C\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{H^2(\{|y|<1\})}\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{L^2(\{|y|<1\})}\\
&\qquad\leq C\left((\omega\epsilon)^2\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{L^2(\{|y|<1\})}+(1+|k|)^{3/2}|J_k(\sqrt{q}\omega\epsilon)|\right)\\
&\qquad\qquad\times\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|_{L^2(\{|y|<1\})} \quad ,
\end{aligned}$$

and so the standard trace theorem implies

$$\begin{aligned}
(1+|k|)|J_k(\sqrt{q}\omega\epsilon)|^2 &\leq C\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{H^{1/2}(\{|y|=1\})}\\
&\leq C\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{H^1(\{|y|<1\})}\\
&\leq C\left((\omega\epsilon)^2+(1+|k|)^2\right)\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{L^2(\{|y|<1\})}\\
&\qquad+\frac{1}{2}(1+|k|)|J_k(\sqrt{q}\omega\epsilon)|^2 \quad .
\end{aligned}$$

We immediately conclude that

$$(1+|k|)|J_k(\sqrt{q}\omega\epsilon)|^2\leq C\left(1+\omega\epsilon+|k|\right)^2\|J_k(\sqrt{q}\omega\epsilon r)e^{ik\theta}\|^2_{L^2(\{|y|<1\})} \quad ,$$

which in combination with (33) leads to

$$|T(q_0,q,k,\omega\epsilon)|^2 \geq c\Im(q)\frac{1+|k|}{(1+\omega\epsilon+|k|)^2}|J_k(\sqrt{q}\omega\epsilon)|^2 \quad ,$$

with the positive constant c independent of q_0, k, ω, and ϵ. This establishes (26). If we suppose that $\Im(q) > 0$ then, after insertion of this estimate into the formula (12),

$$\begin{aligned}|\beta_k| &\leq C|a_k|\frac{1+\omega\epsilon+|k|}{\omega\epsilon}\frac{1}{\sqrt{(1+|k|)}}|J_k(\sqrt{q}\omega\epsilon)|^{-1} \\ &\leq C|a_k|\sqrt{1+|k|}\ |J_k(\sqrt{q}\omega\epsilon)|^{-1} \quad ,\end{aligned}$$

for $1 < \omega\epsilon$, and any $k \in \mathbb{Z}$. The estimate for α_k follows directly from this estimate and the formula

$$\alpha_k = (\beta_k J_k(\sqrt{q}\omega\epsilon) - a_k J_k(\sqrt{q_0}\omega\epsilon))\,\frac{1}{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)} \quad . \tag{34}$$

$\square$

Remark 3. *The bounds (27) and (28) complement the estimates (20). The particular k-dependence of the bounds (27) and (28) makes them well suited for estimates of solutions involving infinitely many Fourier coefficients. However, it should be noted that, for a fixed k, (27) and (28) provide estimates in $\omega\epsilon$ that are slightly weaker (by a factor of $\sqrt{\omega\epsilon}$) than the estimates (20).*

Remark 4. *Using techniques entirely similar to those that led to (33) we could alternatively obtain*

$$|T(q_0,q,k,\omega\epsilon)| \geq \frac{2}{\pi\omega\epsilon}\frac{|J_k(\sqrt{q}\omega\epsilon)|}{|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)|} \quad .$$

In combination with (12) and (34) this gives

$$|\beta_k|\ |J_k(\sqrt{q}\omega\epsilon)| \leq |a_k|\ |H_k^{(1)}(\sqrt{q_0}\omega\epsilon)| \quad , \qquad \text{and} \quad |\alpha_k| \leq 2|a_k| \quad , \tag{35}$$

without any assumption that $\Im(q)$ be strictly positive. These alternative estimates are direct generalizations of (20). Furthermore the very simple second estimate, $|\alpha_k| \leq 2|a_k|$, in combination with the earlier proof of Proposition 1 (for $\Im(q) > 0$) immediately verifies that proposition also when $\Im(q) = 0$. The estimates (35) are not very useful for solutions with infinitely many Fourier coefficients.

We are now able to prove an estimate of $u_\epsilon^{(s)}$ for incident waves with an infinite number of Fourier coefficients (a plane wave, for example) similar to that conjectured in Remark 2. As mentioned earlier it is convenient to introduce the discrete Sobolev norms

$$|a|_{h^\sigma} := \left(\sum_{k\in\mathbb{Z}} |a_k|^2 (1+|k|)^{2\sigma} \right)^{1/2} \quad ,$$

$\sigma \in \mathbb{R}$, to measure the asymptotic behaviour of the sequence $a = \{a_k\}_{k=-\infty}^{\infty}$.

Proposition 2. *Suppose q_0 and q satisfy (1), and suppose in addition $\Im(q) > 0$. Let $u^{(inc)}$ be a smooth incident wave given by*

$$u^{(inc)}(r,\theta) = \sum_{k=-\infty}^{\infty} a_k J_k\left(\sqrt{q_0}\omega r\right) e^{ik\theta},$$

for some coefficients a_k. For $\omega\epsilon > 0$ let $u_\epsilon^{(t)}$ and $u_\epsilon^{(s)}$ be the solutions to the following two dimensional scattering problem

$$\begin{aligned} \Delta u_\epsilon^{(t)} + q\omega^2 u_\epsilon^{(t)} &= 0 \quad \text{for } r = |y| < \epsilon \quad , \\ \Delta u_\epsilon^{(s)} + q_0\omega^2 u_\epsilon^{(s)} &= 0 \quad \text{for } r = |y| > \epsilon \quad , \end{aligned}$$

with the transmission conditions

$$\begin{aligned} \partial_r u_\epsilon^{(s)}\Big|_{r=\epsilon} &= \partial_r u_\epsilon^{(t)}\Big|_{r=\epsilon} - \partial_r u^{(inc)}\Big|_{r=\epsilon} \quad , \\ u_\epsilon^{(s)}\Big|_{r=\epsilon} &= u_\epsilon^{(t)}\Big|_{r=\epsilon} - u^{(inc)}\Big|_{r=\epsilon} \quad , \end{aligned}$$

and the outgoing radiation condition

$$\partial_r u_\epsilon^{(s)} - i\omega\sqrt{q_0} u_\epsilon^{(s)} = o(r^{-1/2}) \quad \text{as } r \to \infty \quad .$$

There exists a constant C, depending on q_0 and q, but independent of ω, ϵ, $\{a_k\}$, and R, such that, for any $R \geq \epsilon$, and any $\sigma \in \mathbb{R}$

$$\left\| u_\epsilon^{(s)}\big|_{r=R} \right\|_{H^\sigma_{per}(0,2\pi)} \leq C|a|_{h^{\sigma+\frac{1}{2}}} \frac{\sqrt{\epsilon}}{\sqrt{R}} \quad \text{for } 1 < \omega\epsilon \quad .$$

Proof. The function $u_\epsilon^{(s)}(y)$, $|y| = r > \epsilon$, is given by

$$u_\epsilon^{(s)}(r,\theta) = \sum_{k=-\infty}^{\infty} \alpha_k H_k^{(1)}\left(\sqrt{q_0}\omega r\right) e^{ik\theta} \quad ,$$

where the coefficients α_k are as before. Due to this representation, the $k = 0$ case of the first bound of (20), and the bound (27) from Lemma 5 we obtain

$$\begin{aligned}
\|u_\epsilon^{(s)}|_{r=R}\|^2_{H^\sigma_{per}(0,2\pi)} &= 2\pi \sum_{k=-\infty}^{\infty} |\alpha_k|^2(1+|k|)^{2\sigma}|H_k^{(1)}(\sqrt{q_0}\omega R)|^2 \\
&= 2\pi|\alpha_0|^2|H_0^{(1)}(\sqrt{q_0}\omega R)|^2 \\
&\qquad +2\pi \sum_{0<|k|} |\alpha_k|^2(1+|k|)^{2\sigma}|H_k^{(1)}(\sqrt{q_0}\omega R)|^2 \\
&\le C|a_0|^2\frac{1}{\omega R} + C\sum_{0<|k|} |a_k|^2(1+|k|)^{1+2\sigma}\frac{|H_k^{(1)}(\sqrt{q_0}\omega R)|^2}{|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)|^2} \\
&\qquad +C\sum_{0<|k|} |a_k|^2(1+|k|)^{2\sigma}|J_k(\sqrt{q_0}\omega\epsilon)|^2\frac{|H_k^{(1)}(\sqrt{q_0}\omega R)|^2}{|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)|^2} \\
&\le C|a_0|^2\frac{1}{\omega R} + C|a|^2_{h^{\sigma+\frac{1}{2}}}\frac{\epsilon}{R} + C|a|^2_{h^\sigma}\frac{\epsilon}{R} \\
&\le C|a|^2_{h^{\sigma+\frac{1}{2}}}\frac{\epsilon}{R} \quad ,
\end{aligned}$$

for $1 < \omega\epsilon$, $\epsilon \le R$. In the next to last estimate we have used the fact that

$$\frac{|H_k^{(1)}(\sqrt{q_0}\omega R)|^2}{|H_k^{(1)}(\sqrt{q_0}\omega\epsilon)|^2} \le \frac{\epsilon}{R} \quad ,$$

for $0 < |k|$, $\epsilon \le R$ (see (24) of the proof of Proposition 1) and the fact that

$$|J_k(\sqrt{q_0}\omega\epsilon)| \le 1 \quad ,$$

since

$$\sum_{k=-\infty}^{\infty} |J_k(\sqrt{q_0}\omega\epsilon)|^2 = 1 \quad .$$

The latter identity follows immediately from the representation formula for a plane wave, (6), and Parseval's identity. $\square$

Since $L^2(0,2\pi) = H^0_{per}(0,2\pi)$, and since $|a|_{h^{\sigma+\frac{1}{2}}} \le C_\sigma |a|_\infty = C_\sigma \max_{k\in\mathbb{Z}} |a_k|$, for $\sigma < -1$, the above proposition has as an immediate consequence:

Corollary 1. *Let the notation and the assumptions be as in Proposition 2. There exists a constant C (depending on q and q_0, but) independent of ω, ϵ, $\{a_k\}$, and R, such that, for any $R \ge \epsilon$*

$$\left\| u_\epsilon^{(s)}|_{r=R} \right\|_{L^2(0,2\pi)} \le C|a|_{h^{1/2}}\frac{\sqrt{\epsilon}}{\sqrt{R}} \quad \textit{for } 1 < \omega\epsilon \quad . \tag{36}$$

Furthermore, given any $\sigma < -1$ there exists a constant C_σ, depending on σ (and q, q_0) but independent of ω, ϵ, $\{a_k\}$, and R, such that, for any $R \geq \epsilon$

$$\left\| u_\epsilon^{(s)}|_{r=R} \right\|_{H^\sigma_{per}(0,2\pi)} \leq C_\sigma |a|_\infty \frac{\sqrt{\epsilon}}{\sqrt{R}} \quad \textit{for } 1 < \omega\epsilon \quad . \tag{37}$$

As stated earlier, we suspect that one has the estimate

$$\left\| u_\epsilon^{(s)}|_{r=R} \right\|_{L^2(0,2\pi)} \leq C |a|_\infty \frac{\sqrt{\epsilon}}{\sqrt{R}} \quad .$$

In other words we suspect that the presence of a stronger norm on the right hand side of (36) (or the presence of a weaker norm on the left hand side of (37)) is merely due to our approach. There is a particular case when this is very simple to verify, namely the case of a hard scatterer. We call an inhomogeneity a hard scatterer if there is no transmitted wave, and the boundary conditions (7c)-(7d) on the boundary of the inhomogeneity are replaced by

$$\tilde{u}_\epsilon^{(s)} + u^{(inc)} = 0 \quad \text{ on } \quad r = |y| = \epsilon \quad .$$

For fixed ϵ this boundary condition formally corresponds to $|q| = \infty$. The "improved" (L^2-) estimate is given by the following proposition. We note that the "hard" boundary condition naturally gives rise to a completely different behaviour as $\omega\epsilon \to 0$.

Proposition 3. *Suppose $0 < q_0$. Let $u^{(inc)}$ be an incident wave given by*

$$u^{(inc)}(r, \theta) = \sum_{k=-\infty}^{\infty} a_k J_k \left(\sqrt{q_0}\omega r\right) e^{ik\theta},$$

for some coefficients a_k, with $|a|_\infty = \sup_{k\in\mathbb{Z}} |a_k| < \infty$. Let $\tilde{u}_\epsilon^{(s)}$ be the solution to the following two dimensional scattering problem

$$\Delta \tilde{u}_\epsilon^{(s)} + q_0\omega^2 \tilde{u}_\epsilon^{(s)} = 0 \quad \textit{for } r = |y| > \epsilon \quad ,$$

with the boundary condition

$$\tilde{u}_\epsilon^{(s)}\Big|_{r=\epsilon} = -\, u^{(inc)}\Big|_{r=\epsilon} \quad ,$$

and the outgoing radiation condition

$$\partial_r \tilde{u}_\epsilon^{(s)} - i\omega\sqrt{q_0}\tilde{u}_\epsilon^{(s)} = o(r^{-1/2}) \quad \textit{as } r \to \infty \quad .$$

There exists a constant C, depending only on q_0, such that for any $R \geq \epsilon$

$$\left\| \tilde{u}_\epsilon^{(s)}|_{r=R} \right\|_{L^2(0,2\pi)} \leq C|a|_\infty \begin{cases} (|\log(\omega\epsilon)| + 1)^{-1} \left| H_0^{(1)}(\omega R) \right| & \textit{for} \quad 0 < \omega\epsilon \leq 1 \quad , \\ \sqrt{\epsilon}/\sqrt{R} & \textit{for} \quad 1 < \omega\epsilon \quad . \end{cases}$$

Proof. The function $\tilde{u}_\epsilon^{(s)}$ has the representation

$$\tilde{u}_\epsilon^{(s)}(r,\theta) = -\sum_{k=-\infty}^{\infty} a_k J_k(\sqrt{q_0}\omega\epsilon)\frac{H_k^{(1)}(\sqrt{q_0}\omega r)}{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)} e^{ik\theta} \quad ,$$

from which it immediately follows that

$$\begin{aligned}
\left\| \tilde{u}_\epsilon^{(s)}\big|_{r=R} \right\|^2_{L^2(0,2\pi)} &= 2\pi|a_0|^2|J_0(\sqrt{q_0}\omega\epsilon)|^2 \left| \frac{H_0^{(1)}(\sqrt{q_0}\omega R)}{H_0^{(1)}(\sqrt{q_0}\omega\epsilon)} \right|^2 \\
&\quad + 2\pi \sum_{|k|>0} |a_k|^2 |J_k(\sqrt{q_0}\omega\epsilon)|^2 \left| \frac{H_k^{(1)}(\sqrt{q_0}\omega R)}{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)} \right|^2 \qquad (38) \\
&\leq 2\pi|a_0|^2|J_0(\sqrt{q_0}\omega\epsilon)|^2 \left| \frac{H_0^{(1)}(\sqrt{q_0}\omega R)}{H_0^{(1)}(\sqrt{q_0}\omega\epsilon)} \right|^2 + 2\pi|a|^2_\infty \frac{\epsilon}{R} \quad .
\end{aligned}$$

For the last inequality we have used that

$$\sum_{|k|>0} |J_k(\sqrt{q_0}\omega\epsilon)|^2 \leq \sum_{k=-\infty}^{\infty} |J_k(\sqrt{q_0}\omega\epsilon)|^2 = 1 \quad , \qquad (39)$$

and that

$$\left| \frac{H_k^{(1)}(\sqrt{q_0}\omega R)}{H_k^{(1)}(\sqrt{q_0}\omega\epsilon)} \right|^2 \leq \epsilon/R \quad , \qquad (40)$$

for any $|k| > 0$ and any $R \geq \epsilon > 0$. The estimate (39) follows immediately from the representation formula for a plane wave, (6), and Parseval's identity. The derivation of (40) was already explained in the proof of Proposition 1. The L^2 estimate of this lemma now follows from (38), by observing that, due to well known asymptotic behaviour of Bessel functions,

$$|J_0(\sqrt{q_0}\omega\epsilon)|^2 \left| \frac{H_0^{(1)}(\sqrt{q_0}\omega R)}{H_0^{(1)}(\sqrt{q_0}\omega\epsilon)} \right|^2 \leq C \begin{cases} \frac{\left|H_0^{(1)}(\omega R)\right|^2}{(|\log(\omega\epsilon)|+1)^2} & \text{for} \quad 0 < \omega\epsilon \leq 1 \quad , \\ 1/(\omega R) \leq \epsilon/R & \text{for} \quad 1 < \omega\epsilon \quad , \end{cases}$$

and by observing that

$$\frac{\epsilon}{R} = \frac{\omega\epsilon}{\omega R} \leq C \frac{\left|H_0^{(1)}(\omega R)\right|^2}{(|\log(\omega\epsilon)| + 1)^2} \quad , \quad \text{for } 0 < \omega\epsilon \leq 1 \quad , \quad \epsilon \leq R \quad .$$

The latter estimate follows from the fact that

$$0 < c \leq \frac{\left|H_0^{(1)}(\omega\epsilon)\right|^2}{(|\log(\omega\epsilon)| + 1)^2} \quad , \qquad \text{for } 0 < \omega\epsilon \leq 1 \quad ,$$

and the fact that $r \to r|H_0^{(1)}(r)|^2$ is an increasing function on $(0,\infty)$, see [11], page 446. □

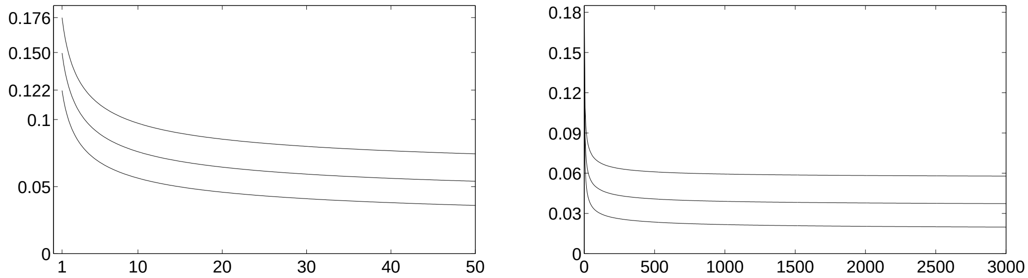


Figure 3: The L^2 norm of $\tilde{u}_\epsilon^{(s)}|_{r=2}$ as a function of ω for three different values of ϵ.

Remark 5. *In Proposition 1 and Proposition 3 the two different cases of the estimate are separated according to whether* $0 < \omega\epsilon \leq 1$*, or* $1 < \omega\epsilon$*. There is of course nothing "sacred" about the number* 1 *in this dichotomy. Due to the continuous dependence of the solution* $u_\epsilon^{(s)}$ *(and* $\tilde{u}_\epsilon^{(s)}$*) on* $\omega\epsilon$*, we could separate according to whether* $0 < \omega\epsilon \leq M$*, or* $M < \omega\epsilon$ *for any fixed* $M > 0$*. The corresponding constants* C_{k_0} *and* C *would depend on* M*.*

Figure 3 illustrates the asymptotic correctness of the estimates in Proposition 3. The two frames display the L^2 norm of $\tilde{u}_\epsilon^{(s)}\big|_{r=2}$, as a function of ω, for three different values of ϵ (.01, .004, and .001). The incident wave on the "hard scatterer" is a plane wave. The left frame shows the clear difference between this case and the case of a soft scatterer (Figure 1) for $\omega\epsilon$ small. The presence of a logarithm in the top estimate in Proposition 3 correctly reflects this difference. The right frame of Figure 3 clearly shows the optimality of the bottom estimate of Proposition 3 as $\omega \to \infty$ for a fixed R.

The techiques used in this paper to prove estimates like those in Proposition 1, Proposition 2, and Proposition 3 are restricted to circular inhomogeneities, due to the heavy reliance on separtion of variables. In the next section we shall very briefly discuss an approach that may partially help extend our results to quite arbitrary, diametrically small convex inhomogeneities. We shall also briefly outline some additional goals of future work.

3 Discussion of extensions and future directions

The proofs of the estimates for the scattering from a soft scatterer and a hard scatterer that were established in the last section differed significantly, the soft scatterer proof being the more involved. There is, however, an approach by which the two cases may be treated in a very similar manner. This approach involves the factorization of the Helmholtz operator inside the inhomogeneity, and the use of part of that factorization to construct an "effective" impedance boundary condition for the scattered field. The approach is worked out in full detail in the theses [4] and [9]. At this place we just briefly describe some of the key elements, and at the same time we indicate some directions for future work.

It is convenient to work with the ϵ-rescaled solution of the original Helmholtz problem (7a)-(7e), in other words to work with

$$U_\lambda^{(t)}(y) = u_\epsilon^{(t)}(\epsilon y) \quad , \text{ and } \quad U_\lambda^{(s)}(y) = u_\epsilon^{(s)}(\epsilon y) \quad ,$$

which form the solutions to

$$\Delta U_\lambda^{(t)} + \lambda^2 q U_\lambda^{(t)} = 0, \qquad \text{for } r = |y| < 1 \quad , \tag{41a}$$

$$\Delta U_\lambda^{(s)} + \lambda^2 q_0 U_\lambda^{(s)} = 0, \qquad \text{for } r = |y| > 1 \quad , \tag{41b}$$

with the following transmission conditions at $r = 1$:

$$\partial_r U_\lambda^{(s)}\Big|_{r=1} = \partial_r U_\lambda^{(t)}\Big|_{r=1} - \partial_r U^{(inc)}\Big|_{r=1} \quad , \tag{41c}$$

$$U_\lambda^{(s)}\Big|_{r=1} = U_\lambda^{(t)}\Big|_{r=1} - U^{(inc)}\Big|_{r=1} \quad , \tag{41d}$$

and the asymptotic radiation condition:

$$\partial_r U_\lambda^{(s)} - i\lambda\sqrt{q_0}U_\lambda^{(s)} = o(r^{-1/2}) \qquad \text{as } r \to \infty \quad . \tag{41e}$$

Here $\lambda = \omega\epsilon$, and

$$U^{(inc)}(r,\theta) = \sum_{k=-\infty}^{\infty} a_k J_k(\sqrt{q_0}\lambda r) e^{ik\theta} \quad .$$

A key result is the following factorization result for the rescaled operator

$$\mathfrak{L}_{q,\lambda} = \Delta + \lambda^2 q \quad , \tag{42}$$

on the annulus $\mathfrak{A} = \{(r,\theta) \ : \ 1/2 < r < 1\}$.

Lemma 6. *Let $\mathfrak{L}_{q,\lambda}$ be the Helmholtz operator given by (42), and suppose $\Im(q) > 0$. There exist nonlocal bounded linear operators $D_q(r,\lambda)$ and $\tilde{D}_q(r,\lambda)$, mapping $H^1_{per}(0,2\pi)$ into $L^2(0,2\pi)$, and given by*

$$D_q(r,\lambda)\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right) = \sum_{k\in\mathbb{Z}} d(k,r,\lambda) c_k e^{ik\theta} \quad ,$$

$$\tilde{D}_q(r,\lambda)\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right) = \sum_{k\in\mathbb{Z}} \tilde{d}(k,r,\lambda) c_k e^{ik\theta} \quad ,$$

such that

$$\mathfrak{L}_{q,\lambda} = \left(\frac{\partial}{\partial r} + \tilde{D}_q(r,\lambda)\right)\left(\frac{\partial}{\partial r} - D_q(r,\lambda)\right) + R_{q,\lambda}$$

where $R_{q,\lambda}$ is a zero order operator in the variables $(r,\theta) \in (\frac{1}{2}, 1) \times (0, 2\pi]$ and in λ , i.e., an operator for which

$$\|R_{q,\lambda} v\|_{L^2(\mathfrak{A})} \leq C\|v\|_{L^2(\mathfrak{A})} \quad ,$$

with C independent of λ.

The proof of this lemma proceeds by separation of variables and fairly straightforward algebraic manipulations The details may be found in [4] or [9]. The formulas derived for the Fourier multipliers $d(k,r,\lambda)$ and $\tilde{d}(k,r,\lambda)$ are

$$d(k,r,\lambda) = \sqrt{\frac{k^2}{r^2} - \lambda^2 q} - \frac{1}{2r}\left(1 - \frac{k^2}{k^2 - \lambda^2 r^2 q}\right) \tag{43}$$

$$\tilde{d}(k,r,\lambda) = \sqrt{\frac{k^2}{r^2} - \lambda^2 q} + \frac{1}{2r}\left(1 + \frac{k^2}{k^2 - \lambda^2 r^2 q}\right) \quad . \tag{44}$$

A careful analysis of these formulas, which is also found in [4] and [9], yields the following result.

Lemma 7. *Suppose $\Im(q) > 0$. The Fourier multipliers $d(k,r,\lambda)$ and $\tilde{d}(k,r,\lambda)$, associated with the operators $D_q(r,\lambda)$ and $\tilde{D}_q(r,\lambda)$ from Lemma 6, satisfy*

$$|d(k,r,\lambda)| \leq C_q(|k| + \lambda) \quad , \quad \textit{and} \quad |\tilde{d}(k,r,\lambda)| \leq C_q(|k| + \lambda) \quad ,$$

for all $1/2 < r < 1$, $k \in \mathbb{Z}$, and $\lambda > 1$. In addition

$$\Re(d(k,r,\lambda)) \geq c_q(|k| + \lambda) \quad , \qquad \Re(\tilde{d}(k,r,\lambda)) \geq c_q(|k| + \lambda) \quad ,$$

and

$$\Im(d(k,r,\lambda)) \leq -C_q \lambda \min\{1, \lambda/|k|\} \quad ,$$

for all $1/2 < r < 1$, $k \in \mathbb{Z}$, and $\lambda > \lambda_q$. The positive constants C_q, c_q and λ_q depend on q, but are independent of k, r, and λ.

Lemma 6 and Lemma 7 yield an estimate for the L^2 norm of $U_\lambda^{(t)}$ on the rescaled inhomogeneity $B = \{y \ : \ |y| < 1\}$, which may then be used to obtain an estimate of $U_\lambda^{(s)}$ (and $u_\epsilon^{(s)}$) outside the inhomogeneity (completely as in the case of a hard scatterer). Very briefly, the estimate for $U_\lambda^{(t)}$ on $B = \{y \ : \ |y| < 1\}$ is obtained by comparing it to (an appropriate extension of) the solution to

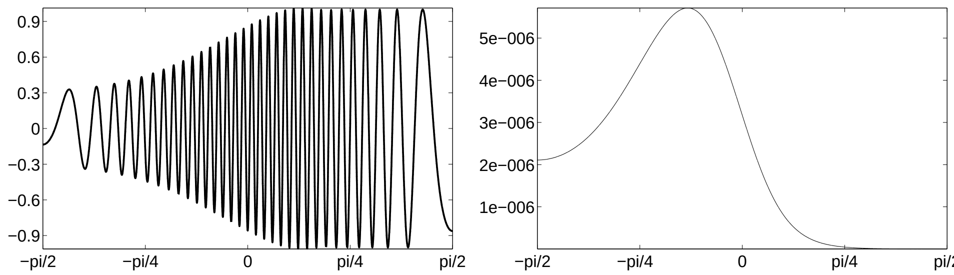
$$\Delta V_\lambda^{(s)} + \lambda^2 q_0 V_\lambda^{(s)} = 0, \qquad \text{for } r = |y| > 1 \quad ,$$

with the following non-local impedance conditions at $r = 1$:

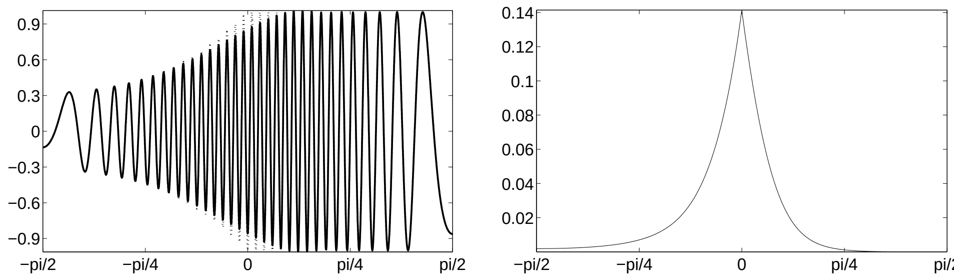
$$\partial_r \left(V_\lambda^{(s)} + U^{(inc)}\right)\Big|_{r=1} - D_q(1,\lambda)\left(V_\lambda^{(s)} + U^{(inc)}\right)\Big|_{r=1} = 0 \quad , \tag{45}$$

and the asymptotic radiation condition :

$$\partial_r V_\lambda^{(s)} - i\lambda\sqrt{q_0} V_\lambda^{(s)} = o(r^{-1/2}) \quad \text{as } r \to \infty \quad .$$



(a) On the left: $\Re(u_\epsilon^{(s)})$ (bold) and $\Re(v_\epsilon^{(s)})$ (dotted). On the right: $|u_\epsilon^{(s)} - v_\epsilon^{(s)}|$. $v_\epsilon^{(s)}$ is the approximation to the scattered field, obtained using the impedance boundary condition.

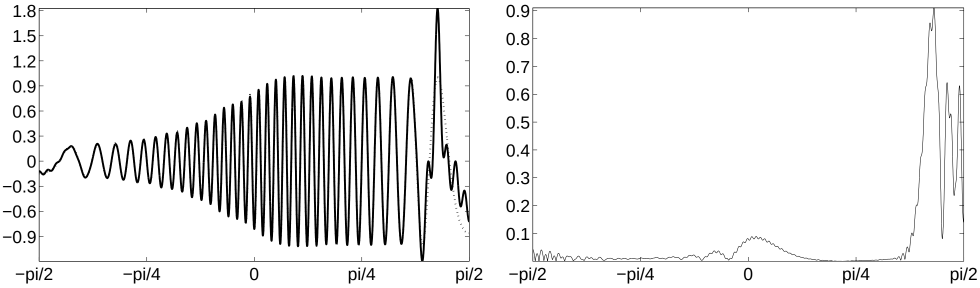


(b) On the left: $\Re(u_\epsilon^{(s)})$ (bold) and $\Re(w_\epsilon^{(s)})$ (dotted). On the right: $|u^{(s)} - w_\epsilon^{(s)}|$. $w_\epsilon^{(s)}$ is the approximation to the scattered field coming from a geometric optics approach.

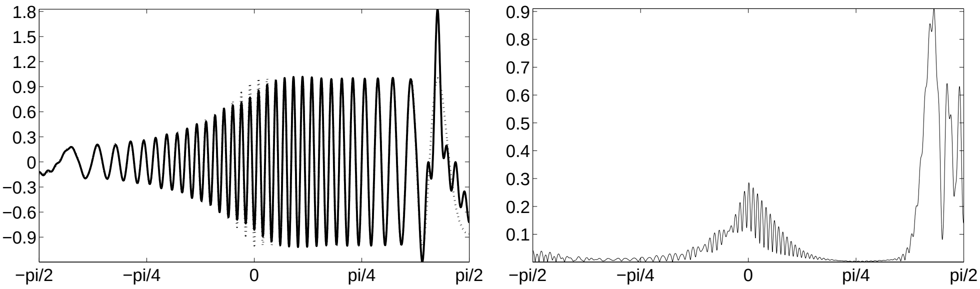
Figure 4: Plots on the right half of the scatterer $r = \epsilon$. The plane wave is incident at the point corresponding to the angle $-\pi/2$. We have chosen $q = 2+2i$, $q_0 = 1$, $\epsilon = 10^{-4}$ and $\omega = 10^6$.

To estimate (the extension of) this solution in terms of the maximum of the Fourier coefficients of the incident wave is relatively simple, in view of the information about the symbol of $D_q(1, \lambda)$, contained in Lemma 7.

Even though the proofs given in [4] and [9] still rely on separation of variables we are fairly confident that this approach may be extended to rather arbitrary convex inhomogeneities through the use of slightly different techniques (see for instance [7]). Such an extension is indeed the subject of current work. One significant goal of future work is also to derive asymptotic representation formulas for the scattered field, that are valid for large (and very broad ranges of) frequencies – much like the ones we have already derived for a single, fixed frequency [10], [2]. It is expected that bounds, such as those derived in this paper, will form a useful element in the construction of these formulas. Using a combination of $\epsilon \to 0$ asymptotics and geometric optics we have already [5] made some progress on formally constructing representation formulas that provide reasonable approximations to the exact solution in the "backscattered region" (and that behave asymtotically as the bounds etablished in this paper). We are hopeful that the impedance boundary condition (45) and the corresponding solution $V_\lambda^{(s)}$ will make it possible to obtain even better approximations to the exact scattered



(a) On the left: $\Re(u_\epsilon^{(s)})$ (bold) and $\Re(v_\epsilon^{(s)})$ (dotted). On the right: $|u_\epsilon^{(s)} - v_\epsilon^{(s)}|$. $v_\epsilon^{(s)}$ is the approximation to the scattered field, obtained using the impedance boundary condition.



(b) On the left: $\Re(u_\epsilon^{(s)})$ (bold) and $\Re(w_\epsilon^{(s)})$ (dotted). On the right: $|u_\epsilon^{(s)} - w_\epsilon^{(s)}|$. $w_\epsilon^{(s)}$ is the approximation to the scattered field coming from a geometric optics approach.

Figure 5: As in Figure 4, but with $q = 2 + i/50$.

solution. Our last figures illustrate this point.

The left frames in Figure 4 show the real part of the scattered solution and two different approximations. All functions are evaluated on the boundary of the inhomogeneity (a conducting circular inhomogeneity with $q = 2 + 2i$). The incident wave is a plane wave, and the scattered solution and its approximations are only shown on half of the boundary, the point of first incidence corresponding to $-\pi/2$. The other half of each graph is obtained by symmetry. In the right frames we display 4(a) the modulus of the difference between the scattered solution $u_\epsilon^{(s)}$ and $v_\epsilon^{(s)} = V_\lambda^{(s)}(\cdot/\epsilon)$ (the rescaled approximation obtained by solving the exterior boundary value problem using the impedance boundary condition (45)) and 4(b) the modulus of the difference between the scattered solution $u_\epsilon^{(s)}$ and its geometric optics approximation $w_\epsilon^{(s)}$ (see [5]). As is evident from these graphs the impedance boundary condition yields a better approximation than the geometric optics approximation. The same phenomenon (albeit less pronounced) is observed even when q has a very small imaginary part, *i.e.*, for an inhomogeneity that is significantly less conducting, as seen in Figure 5.

The ultimate goal is of course to apply the representation formulas that we may derive as a tool to find information about the inhomogeneitites (for instance their location and size) from information about the farfield scattered data. In that connection we hope to be able to use some of the ideas of the direct methods, derived in [1] for the zero frequency case.

References

[1] M. Bruehl, M. Hanke, and M.S. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities. *Num. Math.* **93** (2003) pp. 635–654.

[2] Y. Capdeboscq, and M.S. Vogelius, A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction. *Math. Modelling Num. Anal.* **37** (2003) pp. 159–173.

[3] D. Colton, and R. Kress, *Integral Equation Methods in Scattering Theory* Krieger Publishing Company, Malabar, Florida, 1992.

[4] D.J. Hansen, Ph.D. Thesis, Rutgers University, January 2007.

[5] D.J. Hansen, and M.S. Vogelius, High frequency perturbation formulas for the effect of small inhomogeneities. Preprint, Rutgers University, Jan. 2006.

[6] R. Kress, and G.F. Roach, Transmission problems for the Helmholtz equation. *J. Math. Phys.* **19** (1978) pp. 1433–1437.

[7] O. Lafitte, and G. Lebeau, Equations de Maxwell et opérateur d'impédance sur le bord d'un obstacle convexe absorbant. *C.R. Acad. Sci. Paris, Sér. I Math.* **316** (1993) pp. 1177–1182.

[8] N.N. Lebedev, *Special Functions and Their Applications.* Dover Publications, New York, 1972.

[9] C. Poignard, Ph.D. Thesis, Université de Lyon I, November 2006.

[10] M.S. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. *Modelisation Mathematique et Analyse Numerique* **34** (2000) pp. 723–748.

[11] G.N. Watson, *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, 1944.

Acknowledgments

This research was partially supported by NSF grants DMS–0307119, and DMS–0604999.